PUTNAM PRACTICE SET 2

PROF. DRAGOS GHIOCA

Problem 1. Let $k \in \mathbb{N}$ and let $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{N}$. We know that $\gcd(a_i, b_i) = 1$ for each $i = 1, \dots, k$. We let M be the least common multiple of the numbers $b_1, \dots, b_k$ and also, we let D be the greatest common divisor of the numbers $a_1, \dots, a_k$. Then prove that the greatest common divisor of all the numbers $\frac{a_i \cdot M}{b_i}$ for $i = 1, \dots, k$ is equal to D.

Solution. After replacing each a_i by a_i/D, we may assume that $\gcd(a_1, \dots, a_k) = 1$ and we have to prove that also $\gcd(a_1 M/b_1, \dots, a_k M/b_k) = 1$. Now, assume there exists some prime number p dividing each $a_i M/b_i$. It cannot be that $p \mid a_i$ for each $i = 1, \dots, k$; so, without loss of generality, assume $p \nmid a_1$. Then $p \mid M/b_1$; hence $\exp_p(b_1) < \exp_p(b_i)$ for some $i = 2, \dots, k$ (where $\exp_p(c)$ is the exponent of the prime p in the prime power decomposition of the integer c). Now, let $i_1 \in \{2, \dots, k\}$ such that $\exp_p(b_{i_1}) = \max_{j=1}^k \exp_p(b_j)$; in particular, $p \mid b_1$, but $p \nmid M/b_{i_1}$. So, p must divide a_{i_1}; however this contradicts our hypothesis that $\gcd(a_{i_1}, b_{i_1}) = 1$, which concludes our proof.

Problem 2. Let $P \in \mathbb{Z}[x]$ be a polynomial of degree $\deg(P) \geq 1$. We let $n(P)$ be the number of all integers k for which $(P(k))^2 = 1$. Prove that $n(P) - \deg(P) \leq 2$.

Solution. We argue by contradiction. We let i_1, respectively i_{-1} be the number of integers k such that $P(k) = 1$, respectively $P(k) = -1$. Clearly, $\max\{i_1, i_{-1}\} \leq d$, where $d := \deg(P)$; so, by our assumption, we must have $\min\{i_1, i_{-1}\} \geq 3$.

Now, for each $k_1, k_{-1} \in \mathbb{Z}$ such that $P(k_j) = j$ for $j \in \{-1, 1\}$, we get that $(k_1 - k_{-1}) \mid 2$ and thus $k_1 - k_{-1} \in \{-2, -1, 1, 2\}$. So, if we order the integers in $P^{-1}(-1)$, respectively in $P^{-1}(1)$ as

$$k_{-1,1} > k_{-1,2} > \cdots > k_{-1,i_{-1}}, \text{ respectively}$$
$$k_{1,1} < k_{1,2} < \cdots < k_{1,i_1},$$

we get that for each $j = 1, \dots, i_1$, we have that $k_{1,j} - k_{-1,1} \in \{-2, -1\}$ because there are at least two other differences $k_{1,j} - k_{-1,\ell}$ for $\ell = 2, \dots, i_{-1}$ which are larger than $k_{1,j} - k_{-1,1}$ (and all such differences are either ± 2 or $\pm - 1$). However, since $i_1 \geq 3$, we cannot have that each $k_{1,j} - k_{-1,1} \in \{-2, -1\}$ for each $j = 1, \dots, i_1$. This contradiction finishes the proof of our result.

Problem 3. Let $a_1, \dots, a_5$ be real numbers such that

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 1.$$

Prove that $\min_{1 \leq i < j \leq 5} (a_i - a_j)^2 \leq \frac{1}{10}$.

Solution. We argue by contradiction and so, assume $\min_{1 \leq i < j \leq 5} (a_i - a_j)^2 > \frac{1}{10}$; without loss of generality, we may assume $a_1 < a_2 < a_3 < a_4 < a_5$. Then we replace $a_1, \dots, a_5$ with numbers $b_1, \dots, b_5$ such that

(1) $b_{i+1} - b_i = \frac{1}{\sqrt{10}}$ for each $i = 1, \dots, 4$.
(2) $\sum_{i=1}^5 b_i^2 < \sum_{i=1}^5 a_i^2 = 1$.

Indeed, if all a_i have the same sign, then without loss of generality (after flipping the sign of each a_i), we may assume $0 \le a_1 < a_2 < a_3 < a_4 < a_5$ and then simply take $b_1 = a_1$ and then $b_{i+1} = b_i + \frac{1}{\sqrt{10}}$ for each $i = 1, \dots, 4$.

Now, if $a_1 < 0 < a_5$, then we let $j \in \{1, 2, 3, 4\}$ such that $a_j < 0 \le a_{j+1}$. Then we simply choose $b_j < 0 \le b_{j+1}$ such that

(1) $b_{j+1} - b_j = \frac{1}{\sqrt{10}}$.
(2) $a_j < b_j < 0 \le b_{j+1} \le a_{j+1}$.

Then we let $b_i = b_j - \frac{j-i}{\sqrt{10}}$ for $i = 1, \dots, j-1$ and also, let $b_i = b_{j+1} + \frac{i-j-1}{\sqrt{10}}$ for $i = j+2, \dots, 5$; clearly, the numbers $b_1, \dots, b_5$ satisfy the above two conditions.

Now, we compute

$$\begin{aligned} &\sum_{i=1}^5 b_i^2 \\ &= b_1^2 + \sum_{i=1}^4 \left(b_1 + \frac{i}{\sqrt{10}}\right)^2 \\ &= 5b_1^2 + 2\sqrt{10} \cdot b_1 + 3 \\ &= 5\left(b_1 + \sqrt{\frac{2}{5}}\right)^2 + 1 \\ &\ge 1 \end{aligned}$$

This contradicts the fact that $\sum_{i=1}^5 b_i^2 < 1$.

Problem 4. Let n be a positive integer. Prove that the number

$$\sum_{k=0}^n \binom{2n+1}{2k+1} \cdot 8^k$$

is not divisible by 5.

Solution. We consider $\left(1+\sqrt{8}\right)^{2n+1} =: u_n + v_n \cdot \sqrt{8}$; we observe that $v_n = \sum_{k=0}^n \binom{2n+1}{2k+1} \cdot 8^k$. We have that $u_n^2 - 8v_n^2 = -7^{2n+1}$ and so, assuming that $5 \mid v_n$, we obtain that

$$u_n^2 \equiv -2^{2n+1} \equiv -2 \cdot (-1)^n \equiv \pm 2 \pmod 5,$$

which is a contradiction since neither 2 nor 3 are squares modulo 5.

Problem 5. Let n be a positive integer, let $a_1, \dots, a_n$ be positive real numbers, and let $q \in (0, 1)$ be a real number. Prove that there exist n real numbers $b_1, \dots, b_n$ satisfying the following properties:

- $a_k < b_k$ for each $k = 1, \dots, n$;
- $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for $k = 1, \dots, n-1$; and
- $b_1 + \cdots + b_n < \frac{1+q}{1-q} \cdot (a_1 + \cdots + a_n)$.

Solution. We have the following inequalities (after repeated applications of the first two conditions above)

$$b_k > qb_{k+1} > q^2 b_{k+2} > \cdots > q^\ell b_{k+\ell} > q^\ell a_{k+\ell}$$

and

$$b_k > qb_{k-1} > q^2 b_{k-2} > \cdots > q^\ell b_{k-\ell} > q^\ell a_{k-\ell}.$$

So, we define an auxilliary sequence

$$c_k = \max_{i=1}^{n} q^{|i-k|} a_k \text{ for each } k = 1, \ldots, n;$$

then we need $b_k > c_k$ for each $k = 1, \ldots, n$. We claim that if we let $b_k = c_k + \epsilon$ for some small positive real number ϵ, then all of the three inequalities above will be satisfied. Indeed, since for each $k = 1, \ldots, n$ (looking individually at each number in the set whose maximum is represented by c_k, respectively by c_{k+1}), we have that

$$q \leq \frac{c_{k+1}}{c_k} \leq \frac{1}{q},$$

then we get that

$$q < \frac{c_{k+1} + \epsilon}{c_k + \epsilon} < \frac{1}{q} \text{ for any } \epsilon > 0$$

because

$$qc_k < c_{k+1} + \epsilon(1-q) \text{ and similarly, } qc_{k+1} < c_k + \epsilon(1-q).$$

Also, from the definition of the c_k, we have $c_k \geq a_k$ and therefore, $b_k > a_k$. Now, for the last inequality, we compute

$$\begin{aligned}
\sum_{k=1}^{n} b_k \\
= n\epsilon + \sum_{k=1}^{n} c_k \\
< n\epsilon + \sum_{k=1}^{n} \sum_{i=1}^{n} q^{|i-k|} a_i \\
< n\epsilon + \left(\sum_{i=1}^{n} a_i \right) \cdot \left(1 + 2q + \cdots + 2q^{n-1}\right) \\
< n\epsilon + \left(\sum_{i=1}^{n} a_i \right) \cdot \left(\frac{2}{1-q} - 1 - 2q^n - 2q^{n+1} - \cdots \right) \\
< \frac{1+q}{1-q} \cdot \sum_{k=1}^{n} a_k - \left(2q^n \sum_{k=1}^{n} a_k - n\epsilon \right) \\
< \frac{1+q}{1-q} \cdot \sum_{k=1}^{n} a_k,
\end{aligned}$$

as long as $\epsilon < \frac{2q^n S}{n}$, where $S := \sum_{k=1}^{n} a_k$. This concludes our proof.

Problem 6. For each $n \in \mathbb{N}$, we let Q_n be a square of side length $\frac{1}{n}$. Prove that in a square of side length $\frac{3}{2}$ we can arrange all the squares Q_n such that for any

$m \neq n$, the squares Q_m and Q_n are placed so that there are no interior common points for both Q_m and Q_n.

Solution. We put Q_1 in the bottom left corner with coordinates $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Then we put Q_2 in the top right corner with coordinates $(1,3/2)$, $(3/2,3/2)$, $(1,1)$ and $(3/2,1)$ and we put Q_3 in the upper left rectangle so that the coordinates of Q_3 are $(0,1)$, $(0,4/3)$, $(1/3,4/3)$ and $(1/3,1)$.

We split the remaining right rectangle (whose coordinates are $(1,0)$, $(3/2,0)$, $(3/2,1)$ and $(1,1)$) into vertical strips of lengths $1/4, 1/8, \ldots, 1/2^n, \ldots$ (and common height 1). Note that the sum of the lengths of these strips is

$$1/4 + 1/8 + \cdots + 1/2^n + \cdots = 1/2.$$

Then we arrange $Q_4, \ldots, Q_7$ in the first strip since their lengths is at most $1/4$ and the sum of their heights is

$$1/4 + 1/5 + 1/6 + 1/7 < 4 \cdot 1/4 = 1.$$

Similarly, in the second strip we arrange $Q_8, \ldots, Q_{15}$ since their lengths is at most $1/8$ and the sum of their heights is

$$1/8 + \cdots + 1/15 < 8 \cdot 1/8 = 1.$$

Always, $Q_{2^n}, Q_{2^n+1}, \cdots, Q_{2^{n+1}-1}$ will fit in the the strip of length $1/2^n$.